Welcome to Cataland: An overview

In Cataland, every type of object is counted by the same sequence of numbers (these are the Catalan numbers). Counting the objects is fun, but the real fun begins when we try to find a natural correspondence (bijection) between the objects. After you have counted the first few examples of the following things, try to find a one-to-one correspondence between different "editions" of Cataland!

Welcome to Cataland: The Money Edition!

You have a money-jar. It starts empty. On the first day, you put a dollar in the jar. Every day after that, you either put a dollar into the jar, or take a dollar out. After an even number of days pass, the jar is empty again! You can record this with a sequence of +1 and -1s. We want to count the number of ways this can happen. We'll keep a record of what we've done that day (put a dollar in or a take a dollar out) as a sequence of +1's and −1's. We'll call such a sequence a *good sequence*.

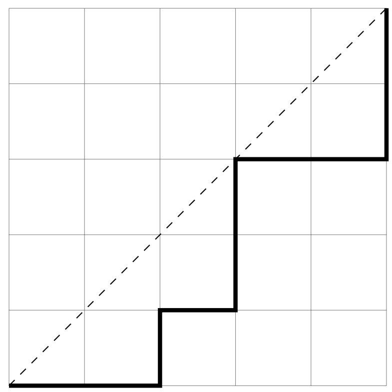
For example, here is a good sequence:

$$+1, -1, +1, +1, +1, -1, +1, -1, -1, -1.$$

How many sequences like this are there that last for 10 days?

Welcome to Cataland: Diagonal-Avoiding Walks Edition!

Imagine you're walking on a grid. You can only make North and East steps, and you must avoid passing above the main diagonal. We call such a walk a *diagonal avoiding walk*. For example,

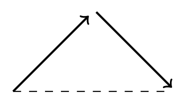


(A) How many diagonal avoiding paths are there to $(5, 5)$?

(B) How many diagonal avoiding paths are there to $(6, 5)$? Diagonal avoiding now means staying below the straight line from $(0, 0)$ to $(6, 5)$.

Welcome to Cataland: Mountain Ranges Edition!

Definition 1 (Mountain Ranges)**.** A mountain range is a sequence of up-strokes ↗ and down-strokes ↘ that start and end on a horizontal line, and never pass below that horizontal line. We will count mountain ranges by the number of strokes they use. (We will always use an even number of strokes.) There?s one mountain range with 2 strokes, and it looks like this:



Problem 1. Draw all of the possible mountain ranges that you can make with 4 strokes. *Hint: The answer is bigger than one, but it's still very small.*

Problem 2. Now repeat the previous problem but this time try to draw

- All of the possible mountain ranges with 6 strokes. *Hint: The answer is still pretty small—there are fewer than 10 ways to do it!*
- All of the possible mountain ranges with 8 strokes. *Hint: There are fewer than 15!*
- All of the possible mountain ranges with 10 strokes. *This is a challenge!*

Welcome to Cataland: The lots of brackets edition!

A valid bracketing is a sequence of 2n brackets, n left and n right, and pair up correctly. For example,

$$()(()(()()))$$

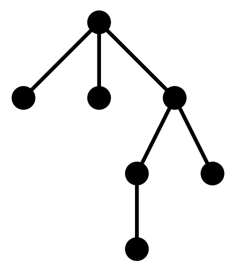
How many valid bracketings are there with exactly 10 brackets?

Welcome to Cataland: Ordered Trees Edition!

An *ordered tree* is a collection of nodes connected by edges so that

- There is a top node.
- Every other node is connected to exactly one node above it, and some number directly below it.
- you can tell the order of the nodes directly below an node by how the picture is drawn (you can mostly ignore this, but that is why they are called “ordered”).

For example, here is an example of such a “tree”.



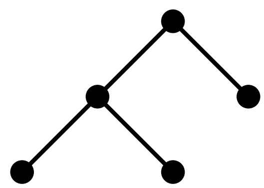
How many different ordered trees are there with 6 vertices?

Welcome to Cataland: PBR Trees Edition!

A *planar binary rooted (PBR) tree* is a collection of vertices connected with edges so that

- There is a "top" vertex which is connected to exactly two vertices below it.
- Every other vertex is connected to exactly one vertex above it, and either two or zero vertices below.
- If a vertex is connected to two vertices below, we keep track of which is on the left and which is on the right (you can mostly ignore this).

For instance, here is a PBR tree:



How many PBR trees are there with 11 vertices?

Welcome to Cataland: Triangulations Edition!

Definition 2 (A triangulation of polygon.)**.** Fix your favorite convex polygon. A *diagonal* is a straight line segment that connects two vertices that are not adjacent.

A *triangulation* is any way to draw as many diagonals as possible through the interior of your polygon so that no two diagonals cross. (The interior of your polygon will look like it's tiled with triangles!) Two triangulations are different if they use a different collection of diagonals.

For example: There's only one way to triangulate a triangle. You can't draw any diagonals because a triangle doesn't have any!

Problem 3. Fix your favorite four sided convex polygon. Draw all of the ways to triangulate this polygon. (Make a new polygon for each triangulation.) How many triangulations are there? *Hint: The answer is small, but it's bigger than 1!*

Problem 4. Now repeat the previous problem but fix your favorite...

- Five sided convex polygon. *Hint: The answer is still pretty small—there are fewer than 10 ways to do it!*
- Six sided convex polygon. *Hint: There are fewer than 15!*
- Seven sided convex polygon. *This is a challenge!*